An Investigation Into Complete (k, l)-Sum-Free Sets

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Abstract

A subset A of a group G is complete (k, l)-sum-free if it has the following two properties. (1) The sets of all possible sums of k and l elements in A are disjoint, and (2) when combined, the set of all possible sums of k and l elements contain every element in G. Our goal is to answer such questions as: "In what groups do complete (k, l)-sum-free sets exist?" among other questions in this topic.

1 Introduction

The following definitions are essential to the contents of this paper.

Definition 1 *The h-fold sumset of a set A is the set of all possible sums of h not necessarily distinct elements in A. This is denoted as hA.*

Definition 2 *We say a set is (k, l)-sum-free if $kA \cap lA = \emptyset$.*

Definition 3 *We say a set is complete (k, l)-sum-free if for a group G, $A \subset G$ $kA \cap lA = \emptyset$* ***and*** *$kA \cup lA = G$.*

Definition 4 *We will define the function $\omega(k, l)$ to be the largest n such that $\mathbb{Z}_n$ has no complete (k, l)-sum-free subset or ∞ if no such largest n exists.*

There are not many previous results in this topic, but I feel this theorem from [1] (which I have summarized and reworded for cyclic groups) is important to the topic.

Theorem 5 (Bajnok 2018) *Let A be a maximum size $(2, 1)$-sum-free subset of $\mathbb{Z}_n$. We have the following:*

1. *If n has a prime divisor congruent to 2 mod 3, then A is complete.*
2. *If n is divisible by 3 but has no prime divisors congruent to 2 mod 3, then A is not complete.*
3. *If n has only prime divisors congruent to 1 mod 3 then for $n > 7$, then there is a complete $(2, 1)$-sum-free set with size $|A|$.*

2 Main results

In this paper there are many major and minor results, the significant begin of which are listed here. With a provided lemma, Peter Francis [2] proved that if $\mathbb{Z}_{n_0}$ contains a complete (k, l)-sum-free subset, then so will $\mathbb{Z}_n$ where n is some multiple of n_0.

Proposition 6 (Francis) *If n_0 divides n and B is complete (k, l)-sum-free in $\mathbb{Z}_{n_0}$, then*

$$A = B + n_0 \cdot \mathbb{Z}_{\frac{n}{n_0}} = \bigcup_{i=0}^{n/n_0} A_i$$

is complete (k, l)-sum-free in $\mathbb{Z}_n$, where $A_i = B + n_0 i$ for each $i \in \mathbb{Z}_{\frac{n}{n_0}}$.

Proposition 7 *If $n \notin \{1, 3, 7, 9\}$ then $\mathbb{Z}_n$ has a complete $(2, 1)$-sum-free set. This gives $\omega(2, 1) = 9$.*

The majority of Proposition 7 follow from Theorem 5, but this Proposition completes the classification of n for which $\mathbb{Z}_n$ has no complete $(2, 1)$-sum-free subset. A second, more original result is as follows

Theorem 8 *If $n \notin \{1, 3, 5, 9, 11, 13, 15, 19, 31\}$ then $\mathbb{Z}_n$ has a complete $(4, 1)$-sum-free set. This gives $\omega(4, 1) = 31$*

This result arises from a combination of constructions as well as computations to verify that the groups listed above do not have any complete $(4, 1)$-sum-free sets. The previous two results are rather specific, but we also have these much more general results.

Theorem 9 *For $k = 2 \mod 4$, and $k > 2$, if*

$$n \geq k^3 + k^2 + 5k + 1$$

then $\mathbb{Z}_n$ has a complete $(k, 1)$-sum-free subset. Or $\omega(k, 1) \leq k^3 + k^2 + 5k$ with the same restrictions on k.

A similar, yet improved result that utilized a vastly different proof method is

Theorem 10 *When*

$$n \geq \left\lfloor \frac{4k(k^2 - 1)}{k - 3} \right\rfloor$$

$\mathbb{Z}_n$ has a complete $(k, 1)$-sum-free set for $k = 0 \mod 4$. Or $\omega(k, 1) \leq \left\lfloor \frac{4k(k^2-1)}{k-3} \right\rfloor - 1$ for $k = 0 \mod 4$.

When combined, we have the summarized result of

Corollary 11

$$\omega(k, 1) \leq \begin{cases} k^3 + k^2 + 5k & k = 2 \mod 4 \\ \left\lfloor \frac{4k(k^2-1)}{k-3} \right\rfloor - 1 & k = 0 \mod 4 \end{cases}$$

The final result I will share in this paper is the complete classification of (k, l)-sum-free intervals.

Theorem 12 *An interval is complete (k, l)-sum-free in $\mathbb{Z}_n$ if and only if*

$$A = \left[\frac{1}{2} \left(\frac{jn}{\gcd(n, k - l)} - \frac{n - 2}{k + l} \right), \frac{1}{2} \left(\frac{jn}{\gcd(n, k - l)} + \frac{n - 2}{k + l} \right) \right]$$

and all the following conditions hold

1. *j is odd*
2. *$\frac{n-2}{k+l}$ is an integer*
3. *n is a multiple of 2^B, where B is the largest non-negative integer such that $\frac{k-l}{2^B}$ is an integer.*
4. *$\frac{n-2}{k+l} = \frac{jn}{\gcd(n,k-l)} \mod 2$*

3 Methods

We will begin with the proof for Proposition 6 that Peter Francis proved with the help of this lemma.

Lemma 13 *If n_0 divides n and*

$$A = \bigcup_{i=0}^{n/n_0} B + n_0 i \subset \mathbb{Z}_n,$$

then for any $l \in \mathbb{N}$,

$$lA = \bigcup_{i=1}^{n/n_0} lB + n_0 i.$$

Proof of Lemma 13. If we take any $a \in lA$, we can find $a_1, \ldots, a_l \in A$ that sum to a, and for each $j \in \{1, \ldots, l\}$, there is some $b_i \in B$ and some $i_j \in \mathbb{Z}_{n/n_0}$ such that $a_j = b_j + i_j n_0$. Then

$$a = \sum_{j=1}^{l} a_j = \sum_{j=1}^{l} b_j + i_j n_0 = \sum_{j=1}^{l} b_j + n_0 \sum_{j=1}^{l} i_j.$$

Note that there is some $i_\alpha \in \mathbb{Z}_{n/n_0}$ for which

$$i_\alpha \equiv \left(\sum_{j=1}^{l} i_j \right) \mod \frac{n}{n_0}, \quad \text{so} \quad n_0 i_\alpha \equiv \left(n_0 \sum_{j=1}^{l} i_j \right) \mod n.$$

Then continuing the first centered equation above,

$$a = \left(\sum_{j=1}^{l} b_j \right) + n_0 i_\alpha \in \bigcup_{i=0}^{n/n_0} lB + n_0 i.$$

Now if we take some $b \in \bigcup_{i=0}^{n/n_0} lB + n_0 i$, we can find $b_1, \ldots, b_l \in B$ and some $i \in \mathbb{Z}_{n/n_0}$ for which

$$b = \left(\sum_{j=1}^{l} b_j\right) + n_0 i = \left(\sum_{j=1}^{l-1} b_j\right) + (b_l + n_0 i) \in lA.$$

■ We make use of this lemma in the proof for Proposition 6 below.

Proof of Proposition 6. Take any $a_1, \ldots, a_k \in A$. Then there exist $i_1, \ldots, i_k \in \mathbb{Z}_{\frac{n}{n_0}}$ and $b_1, \ldots, b_k \in B$ such that for each $j \in \{1, \ldots, k\}$,

$$a_j = b_j + i_j n_0 \in A_{i_j}.$$

Since B is (k,l)-sum-free,

$$\left(\sum_{j=1}^{k} a_j\right) \bmod n_0 = \left(\sum_{j=1}^{k} b_j\right) \bmod n_0 \notin lB,$$

so $\left(\sum_{j=1}^{k} a_j\right) \notin lA$ by Lemma 13; thus $kA \cap lA = \emptyset$.

Now take any $g \in \mathbb{Z}_n \setminus lA$. Then there exist unique $q \in \mathbb{Z}_{\frac{n}{n_0}}$ and $w \in \mathbb{Z}_{n_0} \setminus B$ for which $g = qn_0 + w$. Since B is complete, we can find $b_1, \ldots, b_k \in B$ such that

$$\sum_{j=1}^{k} b_j \equiv w \mod n_0.$$

Then

$$\sum_{j=1}^{k} b_j \in kA \quad \text{and} \quad \left(\sum_{j=1}^{k-1} b_j\right) + (b_k + yn_0) = g,$$

for some $0 \leq y \leq q$, so A is complete. ■

3.1 Specific values of k and l

We now move on to our proof of the rather simple Proposition 7.

Proof of Proposition 7. Consider the subset, A, of $\mathbb{Z}_n$ where n is odd and divisible by 3. Let $6q+3 = n$ and $A = [q, 2q-1] \cup [4q+4, 5q+3]$ Note that unless

$4q - 2 < 2q + 5$ ($n < 24$) then A is both complete and sum-free. Furthermore, $\mathbb{Z}_{21}$ and $\mathbb{Z}_{15}$ have complete $(2, 1)$-sum-free sets $\{3, 4, 5, 16, 17, 18\}$ and $\{1, 4, 6, 9, 11, 14\}$ respectively. When combined with Theorem 5 and computer verification, we find that $1, 3, 7$ and 9 are the only n for which $\mathbb{Z}_n$ has no complete $(2, 1)$-sum-free subset completing our proof. ■

Next on the list is the proof of Theorem 8.

Proof of Theorem 8 First, note that because $\{1\}$ is complete $(4, 1)$-sum-free in $\mathbb{Z}_2$, by Proposition 6 we only need to consider odd n. for which $\mathbb{Z}_n$ may not have a complete $(4, 1)$-sum-free set.

We have the following constructions:

1. $A = [2q + 1, 3q + 1]$ is complete $(4, 1)$ sum-free for $n = 5q + 2$.
2. $A = [2q + 3, 3q] \cup \{2q + 1, 3q + 2\}$ is complete $(4, 1)$-sum-free for $n = 5q + 3$ when n is odd with the exception of $n = 3$ and $n = 13$.
3. $A = [2q + 4, 3q + 3] \cup [7q + 6, 8q + 5]$ is complete $(4, 1)$-sum-free for $n = 10q + 9$ with the exception of $n = 9$ and $n = 19$.
4. $A = [2q + 4, 3q + 1] \cup [7q + 3, 8q - 3]$ is complete $(4, 1)$-sum-free for $n = 10q + 1$ with the exception of $n = 1$, $n = 11$, $n = 21$, and $n = 31$ however 21 is divisible by 7 which does have a complete $(4, 1)$-sum-free set, meaning it too has one.
5. $A = [2q + 4, 3q + 2] \cup [7q + 3, 8q + 1]$ is complete $(4, 1)$-sum-free for $n = 10q + 5$ with the exception of $n = 5$ and $n = 15$

The culminations of these constructions and checking these groups via computer program complete our proof. ■

3.2 Upper Bounds on $\omega(k, 1)$

The rest of the results presented have much more rigorous proofs, beginning with Theorem 9.

Proof of Theorem 9. Let k be a natural number congruent to 2 mod 4. First, note that because $\{1\}$ is complete $(k, 1)$-sum-free for even k, we only need to consider odd values of n.

Let the set A, a subset of $\mathbb{Z}_n$ be defined as

$$A = \left[\frac{n-2q-1-\mathcal{W}}{4}, \frac{n+2q-\mathcal{W}k+\mathcal{W}-5}{4}\right] \cup \left[\frac{3n-2q+\mathcal{W}k-\mathcal{W}+5}{4}, \frac{3n+2q+1+\mathcal{W}}{4}\right]$$

Where $n = 2q(k+1) + k + \mathcal{W} - 1$, where $\mathcal{W}$ is an even non negative integer, $k = 2 \mod 4$, and q is an integer.

$$kA = \bigcup_{h=0}^{k} \left[h\tfrac{n-2q-1-\mathcal{W}}{4} + (k-h)\tfrac{3n-2q+\mathcal{W}k-\mathcal{W}+5}{4}, h\tfrac{n+2q-\mathcal{W}k+\mathcal{W}-5}{4} + (k-h)\tfrac{3n+2q+1+\mathcal{W}}{4}\right]$$

$$kA = \bigcup_{h=0}^{k} \left[-h\frac{2n+6+\mathcal{W}k}{4} + k\frac{3n-2q+\mathcal{W}k-\mathcal{W}+5}{4}, -h\frac{2n+6+\mathcal{W}k}{4} + k\frac{3n+2q+1+\mathcal{W}}{4}\right]$$

Note that the difference between the upper and lower bounds of each of the intervals kA is made up of is $\frac{4qk+2k\mathcal{W}-4k-k^2\mathcal{W}}{4}$. For two intervals $x = [L_x, U_x]$ and $y = [L_y, U_y]$ where $U_x - L_x = U_y - L_y = D$, if $D + 1 \geq U_x - L_y \geq -1$, then $x \cup y = [L_x, U_y]$. Keeping this property in mind, if we take the difference of the upper bound of the term in the big union for kA when $h = t + 2$ and the lower bound when $h = t$ as follows

$$-(t+2)\frac{2n+6+\mathcal{W}k}{4} + k\frac{3n+2q+1+\mathcal{W}}{4} - \left(-t\frac{2n+6+\mathcal{W}k}{4} + k\frac{3n-2q+\mathcal{W}k-\mathcal{W}+5}{4}\right)$$

$$= \frac{4qk - 4k - k^2\mathcal{W} - 12}{4}$$

So when $\frac{4qk+2k\mathcal{W}-4k-k^2\mathcal{W}}{4} + 1 \geq \frac{4qk-4k-k^2\mathcal{W}-12}{4} \geq -1$, $(q \geq \frac{k^2\mathcal{W}+8}{4k} + 1)$ we can inductively conclude that all the terms in the union for kA with the same parity create one large sequence, meaning

$$kA = \left[-k\tfrac{2n+6+\mathcal{W}k}{4} + k\tfrac{3n-2q+\mathcal{W}k-\mathcal{W}+5}{4}, k\tfrac{3n+2q+1+\mathcal{W}}{4}\right] \cup \left[-(k-1)\tfrac{2n+6+\mathcal{W}k}{4} + k\tfrac{3n-2q+\mathcal{W}k-\mathcal{W}+5}{4}, -\tfrac{2n+6+\mathcal{W}k}{4} + k\tfrac{3n+2q+1+\mathcal{W}}{4}\right]$$

$$kA = \left[\frac{n+2q-\mathcal{W}k+\mathcal{W}-5}{4} + 1, \frac{3n-2q+\mathcal{W}k-\mathcal{W}+5}{4} - 1\right] \cup \left[\frac{3n+\mathcal{W}+2q+1}{4} + 1, \frac{n-2q-\mathcal{W}-1}{4} - 1\right]$$

It is now clear that A is complete $(k, 1)$-sum-free in $\mathbb{Z}_n$ if $q \geq \frac{k^2\mathcal{W}+8}{4k} + 1$ or $q \geq \frac{kW}{4} + 1$ for $k > 2$.

Because the case for $k = 2$ has already been fully solved, we will only consider the case when $k > 2$. Note that $\mathcal{W}$ need not be greater than $2k$, as subtracting $2k+2$ from $\mathcal{W}$ and adding 1 to q will not alter the value of n. This means if we substitute q for $\frac{kW}{4} + 1$ and $\mathcal{W}$ for $2k$ in the formula $n_0 = 2q(k+1) + k + \mathcal{W} - 1$ we can be sure that when every n greater than or equal to n_0 has a complete $(k,1)$-sum-free subset and upon substitution we have

$$n_0 = k^3 + k^2 + 5k + 1$$

which completes our proof. ■

Our next result is very similar to Theorem 9 but the technique is very different.

We will first use the following Lemma to assist us

Lemma 14 *If a set of the form $A = [a,b] \cup [-b,-a]$ has the following properties for a given positive integer n and $k = 0 \mod 4$,*

1. $\frac{n}{4} < b < \frac{kn}{4k-4}$, $1 < a < b$
2. $n - a - 2b \geq (ka)$
3. $0 \geq n - 1 - 2a - 2b$
4. $kn = 4kb - 4a + 4$

Then A is complete $(k,1)$-sum-free in $\mathbb{Z}_n$

Proof of Lemma 14. Let $A = [a,b] \cup [-b,-a]$ be a set in $\mathbb{Z}_n$, with $a < b$. Let A be complete $(k,1)$-sum-free in $\mathbb{Z}_n$ for $k = 0 \mod 4$. Note that $b < \frac{kn}{4k-4}$, because if $b \geq \frac{kn}{4k-4}$ then $0 \in (k-1)A$, which would mean $A \subset kA$, making A not sum-free. Furthermore, we will assume that $b \geq \frac{n}{4}$. We can write kA as follows:

$$kA = [ka,kb] \cup [ka-a-b, kb-b-a] \cup [ka-2a-2b, kb-2b-2a] \cup \cdots \cup [a-bk+b, b-ka+a] \cup [-kb,-ka]$$

Now, assume that every other interval (but not consecutive ones) in the representation for kA about intersects, and be cause k is even, we have that

$$kA = [-kb,kb] \cup [a-bk+b, kb-b-a]$$

The conditions for every other interval to intersect are

$$(kb) - (ka) + 1 \geq (kb - 2a - 2b) - (ka) \geq -1$$

for consecutive ones to not intersect we must have that

$$(kb) - (ka) + 1 \ngeq (kb - a - b) - (ka) \ngeq -1$$

. and for the set to be complete sum-free it must be true that $kb + 1 = a \bmod n$ and

$$n = (kb) - (-kb) + n + 1 + 2b - 2a + 2 + (kb - b - a) - (a - bk + b) + 1$$

(A quantity in parentheses means it is taken mod n) Going condition by condition,

$$(kb) - (ka) + 1 \geq (kb - 2a - 2b) - (ka) \geq -1$$

becomes $(kb-2a-2b)-(ka) \geq -1$ and $(kb)-(ka)+1 \geq (kb-2a-2b)-(ka)$ which are equivalent to $n-a-2b \geq (ka)$ and $0 \geq n-1-2a-2b$. Second, one of $(kb)-(ka)+1 < (kb - a - b) - (ka)$ **or** $(kb - a - b) - (ka) < -1$ must hold, and the first of the pair simplifies to $a+b < n+1$, which is clearly always true, so we can ignore this condition. The Condition $n = (kb) - (-kb) + n + 1 + 2b - 2a + 2 + (kb - b - a) - (a - bk + b) + 1$ is much more complex, however. The proscess for it's simplification is below

$$n = (kb) - (-kb) + n + 1 + 2b - 2a + 2 + (kb - b - a) - (a - bk + b) + 1$$

$$0 = (kb) - (-kb) + 2b - 2a + (kb - b - a) - (a - bk + b) + 4$$

$$0 = (kb) - (-kb) + 2b - 2a + (kb) - b - a + n - (bk) - a - b + n + 4$$

$$0 = 2(kb) - 2(-kb) - 4a + 4 + 2n$$

(Parentheses no longer mean mod n)

$$0 = -2\left(-kb - n\left\lfloor\frac{-kb}{n}\right\rfloor\right) - 4a + 4 + 2n$$

$$2n\left\lfloor\frac{kb}{n}\right\rfloor - 2n\left\lfloor\frac{-kb}{n}\right\rfloor - 2n = -4a + 4 + 4kb$$

$$2n\left\lfloor\frac{kb}{n}\right\rfloor + 2n\left\lceil\frac{kb}{n}\right\rceil - 2n = -4a + 4 + 4kb$$

$$4n\left\lfloor\frac{kb}{n}\right\rfloor = -4a + 4 + 4kb$$

Furthermore, since we have that $\frac{n}{4} \leq b < \frac{kn}{4k-4}$, we have

$$\left\lfloor\frac{kn}{4}\right\rfloor \leq \left\lfloor\frac{kb}{4}\right\rfloor \leq \left\lfloor\frac{k^2 n}{4kn - 4n}\right\rfloor$$

$$\frac{k}{4} \leq \left\lfloor\frac{kb}{4}\right\rfloor \leq \left\lfloor\frac{k}{4}\frac{k}{k-1}\right\rfloor$$

Because $k \geq 4$ we have

$$\frac{k}{4} \leq \left\lfloor \frac{kb}{4} \right\rfloor \leq \frac{k}{4}$$

Which makes our final condition,

$$kn = 4kb - 4a + 4$$

Which means we have proven our lemma (Note, $kb + 1 = a \bmod n$ is implied by this condition, as $k = 0 \mod 4$). ■

From Lemma 14 we can now prove Theorem 10

Proof of Theorem 10. Taking our conditions from Lemma 14, we have that if a set of the form $A = [a, b] \cup [-b, -a]$ has the following properties for a given integer n and $k = 0 \mod 4$,

1. $\frac{n}{4} < b < \frac{kn}{4k-4}$, $1 < a < b$
2. $n - a - 2b \geq (ka)$
3. $0 \geq n - 1 - 2a - 2b$
4. $kn = 4kb - 4a + 4$

Then A is complete $(k, 1)$-sum-free in $\mathbb{Z}_n$ We will assume that $a \geq \frac{n}{4}$, and because of that our second condition above becomes

$$b \leq \frac{n - a - ka + \frac{kn}{4}}{2}$$

. This guarantees condition three to hold as well, and condition one will hold as long as $\frac{n-a-ka+\frac{kn}{4}}{2} > \frac{kn}{4k-4}$ or $a < \frac{n}{4}\frac{k^2+k-4}{k^2-1}$. This leaves us with the two conditions

1. $\frac{n}{4}\frac{k^2+k-4}{k^2-1} > a \geq \frac{n}{4}$
2. $kn = 4kb - 4a + 4$

Because there is always an integer solution to $kn = 4kb - 4a + 4$ every k integer values of a (b increases by 1, and a increases by k), and the gap between $\frac{n}{4}\frac{k^2+k-4}{k^2-1}$

and $\frac{n}{4}$ increases for every n we have that if for an integer n_0 $\frac{n_0}{4}\frac{k^2+k-4}{k^2-1} - \frac{n_0}{4} \geq k$ then for all $n \geq n_0$, n will have a complete $(k, 1)$-sum-free subset

$$\frac{n_0}{4}\frac{k^2+k-4}{k^2-1} - \frac{n_0}{4} \geq k$$

Which simplifies to when $k = 0 \bmod 4$ and

$$n \geq \left\lfloor \frac{4k(k^2-1)}{k-3} \right\rfloor$$

$\mathbb{Z}_n$ has a complete $(k, 1)$-sum-free set, completing our proof. ■

3.3 Complete (k, l)-Sum-Free Intervals

Finally, we have the case for (k, l)-sum-free intervals. Before the proof for Theorem 12, we will first prove another Lemma.

Lemma 15 *For any (k, l)-sum-free interval $A = [x, y] \in \mathbb{Z}_n$*

1. $y + x = 0 \mod \frac{n}{\gcd(n, k-l)}$
2. $(y - x)(k + l) + 2 = n$

Proof of Lemma 15. Let $A = [x, y]$ be a complete (k, l) sum-free interval in $\mathbb{Z}_n$.

1. Because A is complete, we have

$$|kA| + |lA| = |\mathbb{Z}_n|$$

$$(ky - kx + 1) + (ly - lx + 1) = n$$

$$k(y - x) + l(y - x) + 2 = n$$

$$(y - x)(k + l) + 2 = n$$

2. WLOG, we can assume that $ly + 1 = kx$ and $ky + 1 = lx$

$$ly + lx - ky = kx$$

$$ly - ky = kx - lx$$

$$(k-l)y = -(k-l)x$$
$$\frac{(k-l)}{\gcd(n,k-l)}y = \frac{-(k-l)}{\gcd(n,k-l)}x \mod \frac{n}{\gcd(n,k-l)}$$
$$y = -x \mod \frac{n}{\gcd(n,k-l)}$$

Finally we have that for an interval $A = [x,y] \in \mathbb{Z}_n$, if A is complete (k,l)-sum-free then $y + x = 0 \mod \frac{n}{\gcd(n,k-l)}$ and $(y-x)(k+l)+2 = n$. ■

Now, we will assume these necessary conditions to be true of a set to find out what other conditions must be met in the proof for Theorem 12.

Proof of Theorem 12. Let n, k and l be integers such that $y + x = 0 \mod \frac{n}{\gcd(n,k-l)}$ and $(y-x)(k+l)+2 = n$ (The necessary conditions for an interval $[x,y] \in \mathbb{Z}_n$ to be complete (k,l) sum-free) From the first equation we have $y = \frac{jn}{\gcd(n,k-l)} - x$,and for some integer j

$$y - x = \frac{jn}{\gcd(n,k-l)} - 2x$$

$$\frac{jn}{\gcd(n,k-l)} - \frac{n-2}{k+l} = 2x$$

Because $2x$ is even, $\frac{n-2}{k+l} = \frac{jn}{\gcd(n,k-l)} \mod 2$.

$$\frac{1}{2}\left(\frac{jn}{\gcd(n,k-l)} - \frac{n-2}{k+l}\right) = x$$

Now if we take the set

$$A = \left[\frac{1}{2}\left(\frac{jn}{\gcd(n,k-l)} - \frac{n-2}{k+l}\right), \frac{1}{2}\left(\frac{jn}{\gcd(n,k-l)} + \frac{n-2}{k+l}\right)\right]$$
$$kA = \left[\frac{k}{2}\left(\frac{jn}{\gcd(n,k-l)} - \frac{n-2}{k+l}\right), \frac{k}{2}\left(\frac{jn}{\gcd(n,k-l)} + \frac{n-2}{k+l}\right)\right]$$
$$lA = \left[\frac{l}{2}\left(\frac{jn}{\gcd(n,k-l)} - \frac{n-2}{k+l}\right), \frac{l}{2}\left(\frac{jn}{\gcd(n,k-l)} + \frac{n-2}{k+l}\right)\right]$$

A is complete (k,l) sum-free iff the following holds for some j (Arithmetic is now mod n):

$$\frac{l}{2}\left(\frac{jn}{\gcd(n,k-l)} + \frac{n-2}{k+l}\right) + 1 = \frac{k}{2}\left(\frac{jn}{\gcd(n,k-l)} - \frac{n-2}{k+l}\right)$$

$$\frac{kn-2k}{2k+2l}+\frac{ln-2l}{2k+2l}+1=\frac{kjn}{2\gcd(n,k-l)}-\frac{ljn}{2\gcd(n,k-l)}$$
$$\frac{kn-2k}{2k+2l}+\frac{ln-2l}{2k+2l}+1=\frac{nj(k-l)}{2\gcd(n,k-l)}$$
$$\frac{kn-2k}{2k+2l}+\frac{ln-2l}{2k+2l}+1=\frac{j\mathrm{lcm}(n,k-l)}{2}$$
$$\frac{kn+ln}{2k+2l}=\frac{j\mathrm{lcm}(n,k-l)}{2}$$
$$\frac{n}{2}=\frac{j\mathrm{lcm}(n,k-l)}{2}$$

Similarly with we have

$$\frac{k}{2}\left(\frac{jn}{\gcd(n,k-l)}+\frac{n-2}{k+l}\right)+1=\frac{l}{2}\left(\frac{jn}{\gcd(n,k-l)}-\frac{n-2}{k+l}\right)$$
$$\frac{n}{2}=-\frac{j\mathrm{lcm}(n,k-l)}{2}$$

These equations will hold if and only if both of the following conditions is true

1. n is a multiple of 2^B, where B is the largest non-negative integer such that $\frac{k-l}{2^B}$ is an integer
2. j is odd.

If and only if, in addition to the conditions from 15, the above two conditions and $\frac{n-2}{k+l}=\frac{jn}{\gcd(n,k-l)} \mod 2$ hold, then the interval A is complete (k,l)-sum-free in $\mathbb{Z}_n$, finishing our proof. ■

However, I would be remiss if I did not outline some of the consequences of Theorem 12.

Corollary 16 *When* $\gcd(n,k-l)=1$, *every complete* (k,l)*-sum-free arithmetic progression in* $\mathbb{Z}_n$ *is symmetric.*

Corollary 17 *There exists no arithmetic progression that is complete* (k,l)*-sum-free when* k *and* l *are both even.*

Proof of Corollary 17. Let k and l be even positive integers. Note that every arithmetic progression in $\mathbb{Z}_n$ that is (k, l)-sum-free must be a dilation of an interval as $k = l \mod 2$

This means for a given $\mathbb{Z}_n$, k and l

$$k + l \mid n - 2$$

(meaning n is even) and

$$\frac{n-2}{k+l} = \frac{n}{\gcd(n, k-l)} \mod 2$$

Note that from here, if $n = 0 \mod 4$, then the left side is odd, and the right side is even, and it its reversed if $n = 2 \mod 4$, and because n must be even, this equality never holds and our proof is completed.

4 Future work

In regards to the unknown in this topic, there is still quite a lot. Below I have outlined a few of the many questions I have.

Conjecture 18 $\omega(3, 1) \neq \infty$

When $k + l = 0 \mod 2$ the values of n that do not have a complete (k, l)-sum-free set are definitely much more common than the $k + l = 1 \mod 2$ case. In fact, there is no complete $(3, 1)$-sum-free set for odd n until $n = 35$ with $\{4, 5, 9, 10, 11, 16\}$. But, due to the exponentially increasing computation time when n get bigger, I have yet to find another odd n not divisible by 35 for which $\mathbb{Z}_n$ has a complete $(3, 1)$-sum-free set. But I conjecture that eventually, such values of n will become more and more common, and eventually there will be a finite value of $\omega(3, 1)$ such that for all $n > \omega(3, 1)$, $\mathbb{Z}_n$ has a complete $(3, 1)$-sum-free subset.

Conjecture 19 $\omega(4, 2) = \infty$

My primary reason of believing this is Corollary 17.

Conjecture 20 *Every complete $(2, 1)$-sum-free set is symmetric.*

This one I am not sure of. I would think that the proof for this would be rather simple, but it has evaded me despite that fact that every complete $(2,1)$-sum-free set that I have seen is symmetric.

Problem 21 *Find the minimum cardinality of a complete $(2,1)$-sum-free set in $\mathbb{Z}_n$.*

I have done relatively little investigation into this topic, but the patterns that appear in these values are quite interesting.

Problem 22 *Find more values or bounds on $\omega(k,l)$.*

One way to accomplish the above task is to see that the bound $\omega(k,1)$ when $k=0$ mod 4 grows much slower than the one I found for $k=2$ mod 4. So it is very realistic to ask

Problem 23 *Find a better upper bound on $\omega(k,1)$ for $k=2$ mod 4 by using the technique used to prove Theorem 10.*

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